Numerical Study of the Localization Length Critical Index in a Network Model of Plateau-Plateau Transitions in the Quantum Hall Effect

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(Received 12 April 2011; published 3 August 2011)

We calculate numerically the localization length critical index within the Chalker-Coddington model of the plateau-plateau transitions in the quantum Hall effect. We report a finite-size scaling analysis using both the traditional power-law corrections to the scaling function and the inverse logarithmic ones, which provided a more stable fit resulting in the localization length critical index $\nu = 2.616 \pm 0.014$. We observe an increase of the critical exponent ν with the system size, which is possibly the origin of discrepancies with early results obtained for smaller systems.

DOI: 10.1103/PhysRevLett.107.066402 PACS numbers: 71.30.+h, 71.23.An, 72.15.Rn

Plateau-plateau transitions in the quantum Hall effect have been one of the most challenging problems in condensed matter physics during the past two decades. It is an interesting example of the localization-delocalization transition in two-dimensional disordered systems, where a quantum critical point appears due to the breaking of time reversal symmetry. One of the important problems in this area of research is the formulation of a quantum field theory describing the transition. The first suggestion in this respect appeared in Ref. [1], where the authors noticed that the presence of the topological term in the nonlinear sigma model formulation of the problem can result in the occurrence of delocalized states under strong magnetic fields.

Later, Chalker and Coddington [2] formulated a phenomenological model of quantum percolation based on a transfer-matrix approach (referred to as the CC model hereafter). The numerical value 2.5 ± 0.5 of the critical index of the Lyapunov exponent (LE) calculated within the CC model (see Ref. [3] for a review) was in good agreement with the experimentally measured localization length index $\nu = 2.4$ in the quantum Hall effect [4]. This success motivated considerable interest in the CC model and stimulated its further investigation until the present day [5–14]. In early studies the continuum limit of the CC model was related to replicas of ordinary spin chains [6], a Hubbard-like model [7], and supersymmetric spin chains [8,9]. In Refs. [10,11], the continuum limit was also related to the conformal field theory of the Wess-Zumino-Witten-Novikov (WZWN) type. Analyzing the representations of the PSL(2|2) conformal field theory, they found one which gives a reasonable value of $16/7 \simeq 2.286$ for the localization length index. Moreover, multifractal scaling indices of the CC model were predicted to depend quadratically on the parameter q of the multifractal analysis within the WZWN model.

Most intriguing developments in the plateau-plateau transition problem were reported later in Refs. [15,16], where the multifractal behavior of the CC model was investigated. In both papers, approximately quartic deviation from the exact quadratic dependence of the multifractal indices on the parameter q, predicted in Refs. [10,11], was observed. The latter suggested that the validity of the supersymmetric WZWN approach to plateau-plateau transitions in the quantum Hall effect is questionable. On the other hand, since the plateau-plateau transitions are of the second order, there is a conformal symmetry at the transition point and there should exist a conformal field theory describing it. Candidate theories could be tested against the experimental data by comparing critical indices. Unfortunately, the precision of the available experimental indices is too low and does not enable us to identify the correct theory. Therefore, comparison to numerically calculated values can be more feasible for this task. However, reliable calculation of the localization length critical index with good precision has been known to be a very challenging task, and there is still little consensus on the obtained values and especially their error bars. This Letter is largely motivated by the demand for such an accurate calculation.

We have carried out numerical calculations of the smallest LE and the corresponding critical index in the CC model taking finite-size effects into account. For this purpose we have used the transfer-matrix method. The final transfer matrix of the CC model is equal to the product of layer transfer matrices $T = \prod_{j=1}^{L} W_1 U_{1j} W_2 U_{2j}$. Each layer transfer matrix $W_1 U_{1j} W_2 U_{2j}$ corresponds to a vertical strip in Fig. 1 that forms $2M \times 2M$ matrices $W_1 U_1$ and $W_2 U_2$ for neighboring columns. Within the chosen parametrization, the matrices W_1 and W_2 are defined as follows:

$$[W_1]_{2n+1,2n+1} = [W_1]_{2n,2n} = 1/t,$$
$$[W_1]_{2n+1,2n} = [W_1]_{2n,2n+1} = r/t,$$
$$[W_2]_{2n-1,2n-1} = [W_2]_{2n,2n} = 1/r,$$
$$[W_2]_{2n-1,2n} = [W_2]_{2n,2n-1} = t/r,$$
$$[W_2]_{1,2n} = [W_2]_{2n,1} = t/r, \qquad n = 1, \ldots, M,$$

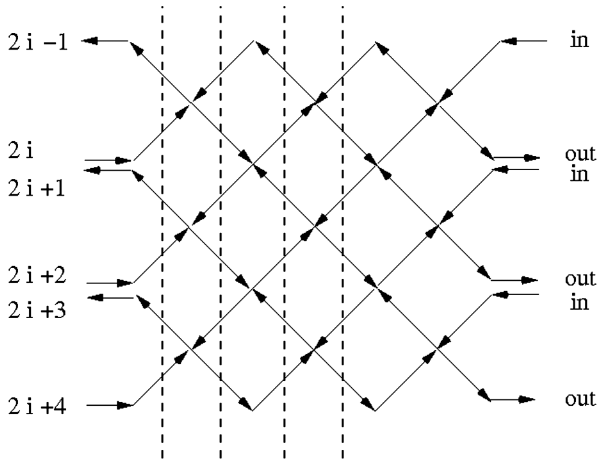

FIG. 1. Schematic view of the CC network.

$t = 1/\sqrt{1 + e^{2x}}$ and $r = \sqrt{1 - t^2}$ being the transmission and the reflection amplitudes, respectively, at each node of the regular lattice shown in Fig. 1. Periodic boundary conditions are imposed on W_2. Matrices U have a simple diagonal form $[U_{1,2}]_{nm} = \exp(i\alpha_n)\delta_{nm}$. The model parameter x corresponds to the Fermi energy measured from the Landau band center scaled by the Landau bandwidth (so the critical point is $x = 0$), while the phases α_n are stochastic variables in the range $[0, 2\pi)$, reflecting the randomness of the smooth electrostatic potential landscape.

To avoid loss of precision during the transfer-matrix multiplication, we performed a modified Gram-Schmidt orthogonalization after every 7 transfer-matrix multiplications [17]. Hereafter, we focus on the smallest LE γ, which can be calculated as

$$\gamma = \lim_{L\to\infty} \frac{\gamma_1}{2L},$$

where γ_1 is the smallest positive eigenvalue of the matrix $\ln(T^\dagger T)$.

Rather than calculating the LEs γ for very long chains, one can obtain them for a large number N_r of shorter chains of length L. Then, by calculating the ensemble average LE $\bar{\gamma}$ and its standard deviation σ_γ and making use of the central limit theorem [18], the error of $\bar{\gamma}$ can be estimated as $\sigma_{\bar{\gamma}} = \sigma_\gamma/\sqrt{N_r}$. Thus, any target precision of $\bar{\gamma}$ can be achieved by increasing N_r. To obtain the desired accuracy, we adjusted N_r for each value of the energy x and the transfer-matrix size M. In our calculation we used a fixed value of $L = 10^6$ with ensemble sizes $N_r \geq 200$ that leads to an effective length of the transfer matrix of $L_{\rm eff} = N_r \times L \simeq 10^8$–$10^9$. By fitting the data, we found that the relative error of the LE obeys the following size scaling: $\sigma_\gamma/\bar{\gamma} \approx \sqrt{M/L}$ as can be seen in Fig. 2.

Having calculated disorder-averaged LEs, we use the standard finite-size scaling analysis of the data, formulated in Ref. [19] and extended in Refs. [20–22], in order to

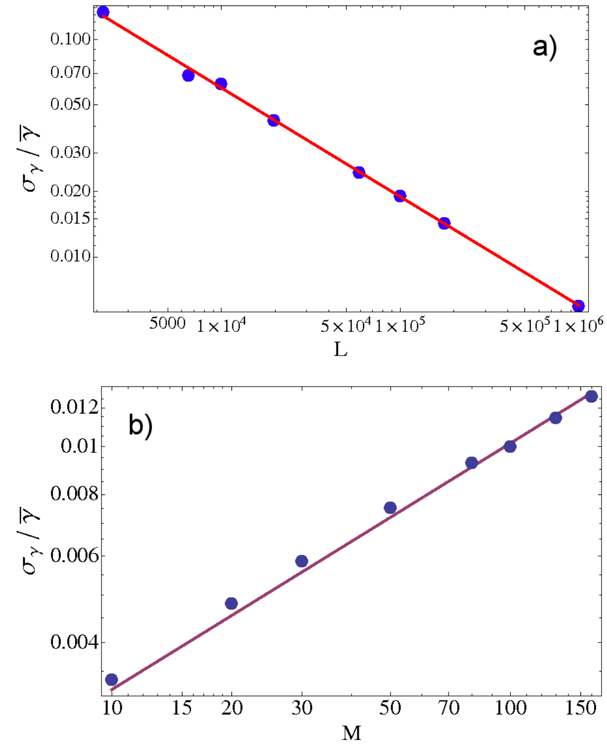

FIG. 2 (color online). Scaling of $\sigma_\gamma/\bar{\gamma}$ as a function of (a) L for $M = 30$ and (b) M for $L = 10^6$ (dots) both at $x = 0.03$. Solid lines show the approximation $\sigma_\gamma/\bar{\gamma} \approx \sqrt{M/L}$.

obtain the localization length index ν. The LE is believed to have scaling behavior (see, for example, Refs. [14,20], and references therein), and finite-size effects can be accounted for by the following formula for the scaling function Γ which approximates $M\bar{\gamma}$ in the vicinity of the critical point:

$$\Gamma = F_0(M^{1/\nu}u_0) + \sum_{k=1}^{n_I} F_k(M^{1/\nu}u_0)[f(M)u_1]^k, \quad (1)$$

where $F_0(\cdot)$, $F_1(\cdot)$, $u_0(\cdot)$, and $u_1(\cdot)$ are universal functions independent of L. The first function $F_0(\cdot)$ is the contribution of the main operator in the corresponding conformal field theory, which defines the localization length. The second function $F_1(\cdot)$ results from the operator with the anomalous dimension which is close to that of the main one and takes into account corrections to the scaling. The function $f(M)$ is a decreasing function of M. Usually, a power-law correction is used: $f(M) = M^y$, where $y < 0$ is the irrelevant exponent. We used the following formula to fit the data [14,22]:

$$\Gamma = \Gamma_c + \sum_{n=1}^{n_R} \alpha_{2n}[u_0 M^{1/\nu}]^{2n} + \sum_{m=1}^{n_I} \beta_m [u_1 f(M)]^m, \quad (2)$$

where $u_0 = \sum_{j=1}^{m_R} a_{2j-1}x^{2j-1}$ with $a_1 = 1$ and $u_1 = 1 + \sum_{i=1}^{m_I} b_i x^{2i}$. The series for F_0, F_1, u_0, and u_1 were truncated at the maximum orders $n_R = 3$, $m_R = 3$, $n_I = 3$, and $m_I = 3$, respectively.

Because all mean LEs have different error bars, a weighted fit was used with $[M\sigma_{\bar{\gamma}}]^{-2}$ as weights. Equation (2) was first fitted to the data with the following constraints: $2.1 \le \nu \le 2.9$, $-1 \le y \le 0$ (for the power-law correction), and $0 < \Gamma_c < 1$, other linear parameters being in the range $(-5, 5)$. The obtained set of the best fit parameters was then used as an initial guess for subsequent unconstrained fits, which were performed to estimate the parameter error bars (see below). All results were characterized by the goodness-of-fit parameter p. To obtain the critical index we considered only the fits with $0.1 \le p \le 0.95$.

In order to estimate the robustness of the results, we fitted our data by the scaling function (2) for all possible combinations of orders n_R and n_I up to the maximum ones and different thresholds $\Gamma_{\max}$ and $M_{\max}$. We did not manage to obtain a stable fit for the power-law correction; the magnitude of the exponent y seemed to be correlated with Γ_c. Moreover, the mean value of the exponent y became smaller than its standard deviation as we increased n_I, suggesting a weaker than power-law dependence of the correction on the system size M. We used the inverse logarithmic corrections: $f(M) = \ln^{-1}(M)$, which yielded robust and consistent results. Although such an ansatz cannot be explained from the common field theory point of view, inverse logarithmic corrections appear to be numerically relevant, which is supported by the robustness of the results.

In Tables I and II, we show the corresponding results for $20 \le M \le 130$ and $20 \le M \le 210$, respectively, calculated for a given order of expansion ($n_R = 2$, $m_R = 1$, $n_I = 3$, and $m_I = 1$) and various $\Gamma_{\max}$. The results of the fits are consistent: They agree well with each other within error bars for all values of $\Gamma_{\max}$ or $M_{\max}$.

The result of the best fit in the case of the inverse logarithmic correction for orders of expansion $n_R = 2$, $m_R = 1$, $n_I = 3$, and $m_I = 1$ is presented in Fig. 3. The curves do not have a common intersection point, which is a signature of finite-size corrections. The crossing point of a pair of the curves corresponding to consecutive values of M shifts towards the origin on increasing M. The latter suggests an additional condition $\beta_1 > 0$, which reflects the increase of the role of the next-to-the-leading operator in the problem for small M. This fact stresses the importance of the second operator together with the main one in the analysis of possible candidate conformal field theories which describe plateau-plateau transitions in the quantum Hall effect.

We used the standard resampling technique [23] to estimate the parameter error bars. For each set of orders of expansion, the model was fitted to 50 synthetic data sets of mean LEs, which were drawn from the corresponding normal distributions centered at $\bar{\gamma}$ and having the standard deviation $\sigma_{\bar{\gamma}}$. From the obtained distributions of the best fit parameters, we calculated their mean values and 95% confidence intervals. The best result was obtained for $n_R = 2$, $m_R = 1$, $n_I = 3$, and $m_I = 1$: $\Gamma_c = 0.702 \pm 0.014$ and $\nu = 2.615 \pm 0.014$.

Our calculated value of the critical index $\nu = 2.615 \pm 0.014$ disagrees with early results [3,24,25], which could be attributed to the quality of the data, system sizes reached, and finite-size effects. Those effects proved to be extremely important for accurate calculation of the critical exponent even in the standard Anderson model [20], in which case the irrelevant exponent $y \sim -3$ is much larger than in the CC model (finite-size effects are therefore less pronounced). On the other hand, our critical exponent is slightly higher than the one calculated recently by Slevin and Ohtsuki [14], who reported $\nu = 2.593 \pm 0.006$. The precision of their data is somewhat better than ours, although our maximum system size is larger. The critical index tends to increase when larger system sizes are considered, as can be seen in Tables I and II. We have studied also the dependence of Γ_c on both the order n_I of the expansion in terms of the irrelevant function and the maximum system size $M_{\max}$ considered. At fixed m_R, m_I, and n_R, the value of Γ_c decreased from 0.772 ± 0.012 for $n_I = 1$ down to 0.66 ± 0.13 for $n_I = 4$ (higher values of n_I led to very low quality of fits). The same trend was observed as $M_{\max}$ was increased.

We note finally that both most recent high precision numerical results are considerably above the experimental value of $\nu = 2.38 \pm 0.06$ measured recently in GaAs-AlGaAs heterostructures [26,27] (the detailed discussion of disagreement with previous results can be found in Ref. [14]). The latter fact emphasizes the necessity of further investigations to clarify the validity of the CC model applied to the plateau-plateau transitions.

In summary, the obtained critical exponent $\nu \approx 2.6$ suggests that the rational value 7/3, which is in agreement with some early calculations, could be questioned (the value 13/5 seems to be much more likely according to

TABLE I. Confidence intervals of the best fit parameters ν and Γ_c obtained for $n_R = 2$, $m_R = 1$, $n_I = 3$, and $m_I = 1$, different upper cutoffs $\Gamma_{\max}$, and the width range $20 \le M \le 130$. Each fit is characterized by its goodness-of-fit parameter p.

$\Gamma_{\max}$	p	ν	Γ_c
1.00	0.95	[2.612, 2.655]	[0.700, 0.730]
1.25	0.39	[2.598, 2.632]	[0.691, 0.724]
1.50	0.56	[2.604, 2.634]	[0.693, 0.725]
2.00	0.18	[2.598, 2.621]	[0.694, 0.728]

TABLE II. The same as in Table I but for $20 \le M \le 210$.

$\Gamma_{\max}$	p	ν	Γ_c
1.00	0.92	[2.616, 2.650]	[0.684, 0.708]
1.25	0.30	[2.603, 2.629]	[0.678, 0.707]
1.50	0.35	[2.608, 2.632]	[0.680, 0.704]
2.00	0.15	[2.604, 2.628]	[0.697, 0.705]

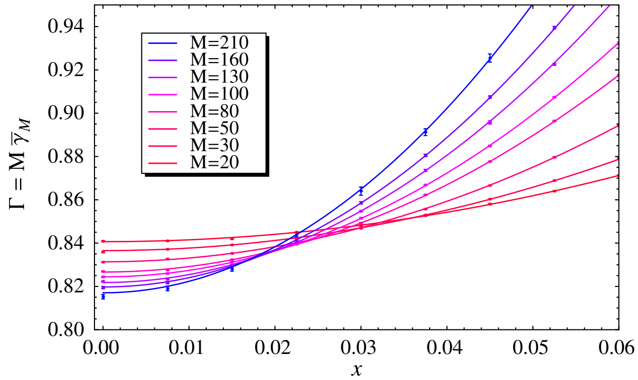

FIG. 3 (color online). Lyapunov indices (dots) with their error bars calculated for different sizes of the transfer matrix and best nonlinear fits (solid lines) to Eq. (2) with $n_R = 2$, $m_R = 1$, $n_I = 3$, and $m_I = 1$.

recent numerical results). Finite-size effects turn out to be very pronounced within the framework of the CC model. We found that inverse logarithmic corrections give a more consistent and robust fit than the traditional power-law corrections. The calculated value of the critical exponent suggests also that some WZWN-type models based on the conformal field theory should be reconsidered, which demands new developments and approaches in the formulation of the continuum limit of the CC model as well as further studies and more accurate calculations of Lyapunov indices.

A. S. acknowledges discussions with I. Gruzberg and V. Kagalovsky and the hospitality of the Universidad Complutense de Madrid, where the major part of the work has been done. A. V. M. and F. D.-A. thank K. Slevin, T. Ohtsuki, and R. Römer for fruitful discussions. Part of the calculations were performed at the Aula Sun Cluster and the Clúster de Cálculo de Alta Capacidad para Técnicas Físicas, funded by the UCM and the EU under the FEDER program. Work in Madrid was supported by MICINN (Projects Mosaico and MAT2010-17180).

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